

Método Espectral Determinístico para Problemas de Transporte de Nêutrons na Formulação S_N , Multigrupo de Energia, Geometria Unidimensional, Espalhamento Anisotrópico e Fonte Fixa.

The Multigroup Spectral Deterministic Method for S_N Neutron Transport Theory in Slab Geometry, Anisotropic Scattering with Fixed-Source Problems

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RESUMO

Neste artigo é apresentada uma nova metodologia numérica contemplando o desenvolvimento de um método da classe dos espectro-nodais (malha grossa) para a solução de problemas de transporte de nêutrons na formulação das ordenadas discretas (S_N), em geometria unidimensional, considerando o espalhamento anisotrópico, fonte fixa e multigrupo de energia. O método, denominado Método Espectral Determinístico (MED), baseia-se, inicialmente, na análise espectral das equações de transporte de nêutrons S_N . As incógnitas dessa metodologia são os fluxos angulares nos contornos e o fluxo angular médio no interior dos nodos espaciais. Os valores numéricos obtidos para essas grandezas, a menos dos erros da aritmética finita computacional, concordam com a solução analítica da equação de transporte S_N no domínio espacial utilizado. Os resultados numéricos são mostrados e comparados com o tradicional método de malha fina, DD, cf., Diamond Difference e os métodos nodais SGF, cf., spectral Green's function e o método F_N para ilustrar a precisão numérica nos resultados obtidos pelo MED.

Palavras-chave: Teoria do transporte de nêutrons; Espalhamento anisotrópico; Metodologia nodal numérica; Ordenadas discretas; Problemas de fonte-fixa; Modelagem computacional determinística; Geometria unidimensional.

ABSTRACT

A new approach for the development of a numerical method of spectral nodal class for the solution of multigroup, anisotropic slab geometry, discrete ordinates transport problems with fixed-source is analyzed in this paper. The method, denominated Spectral Deterministic Method (SDM), is based on the spectral analysis of the neutron transport equations in the formulation of discrete ordinates (S_N). The unknowns in the methodology are the cell-edge, and cell average angular fluxes, the numerical values computed for these quantities concur with the analytic solution of the discrete ordinate's equation. Numerical results are given and compared with the traditional fine-mesh DD method, the spectral nodal method, spectral Green's function (SGF) and the F_N method to illustrate the method's numerical accuracy.

Keywords: Multigroup neutron transport theory; Anisotropic scattering; Numerical nodal methodology; Fixed-source discrete ordinates problems; Computational modeling; One-dimensional slab geometry.

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1. INTRODUCTION

Several searches (GARCIA and SIEWERT, 1983; LAWRENCE, 1986; BARROS, 1990; BARROS and LARSEN, 1990; ABREU, 1996; YAVUZ, 2014) have been performed on the accurate (numerical) solution of the coarse-mesh methods for Cartesian geometry discrete ordinates (S_N) transport problems, which approximates the linear Boltzmann transport equation describing the transport of neutrons in a nuclear reactor (DUDERSTADT and MARTIN, 1979; LEWIS and MILLER, 1993), and in radiation shields problems. These researches and some others have made possible the development of deterministic numerical methods for obtaining accurate computational solutions to the radiation shields problems, global reactor calculations, and other applications.

A primary goal of computational neutron transport is the reliable prediction of neutron production and loss rates (DUDERSTADT and MARTIN, 1979; LEWIS and MILLER, 1993). In this context, were developed methods like the fine-mesh method *DD* (LEWIS and MILLER, 1993), the MOC, c.f., Method of Characteristics (ASKEW, 1972; JEVREMOVIC et al., 2001; MAZUMDAR and DEGWEKER, 2015) and the coarse-mesh *SGF* (BARROS, 1990; BARROS and LARSEN, 1990), as well as others (SILVA et al., 2013). These numerical methods make possible nuclear computer modeling using a deterministic approach. They usually, less the MOC, use the formulation of discrete ordinates (S_N) (LEWIS and MILLER, 1993) and the multigroup approximation to discretized the angular and the energy variables in the transport equation. The multigroup formulation is an approximation of the energy-dependent transport equation in which the energy variable is discretized into contiguous groups. The S_N discrete ordinates scheme in slab geometry problems made a discretization of the angular variables in N directions (discrete ordinates) and used an angular quadrature set for the approximation of the integral source terms. The foundations of the *DD* method are given by the linear approximation of the neutron angular flux, where the value of the average angular flux at each node is the arithmetic mean of the angular fluxes at the spatial cell interfaces (LEWIS and MILLER, 1993). The *SGF* method, introduced by R.C. Barros and E.W. Larsen (BARROS, 1990; BARROS and LARSEN, 1990) in 1990; is a numerical method that is free from spatial truncation errors for a general one-group, slab geometry, discrete ordinates problems with linearly anisotropic scattering and a prescribed internal source. This method is based on the use of the neutral particle balance equations, together with a non-standard auxiliary equation that contains a Green's function (BARROS and

LARSEN, 1990). The unknowns are the cell-edge and cell average angular fluxes, the numerical values computed for these quantities are the same as the analytic solution of the discrete ordinate's equation (BARROS and LARSEN, 1990).

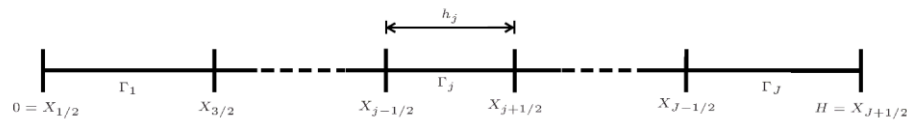
In an earlier paper (OLIVA et al., 2016; OLIVA et al., 2018), a numerical method was proposed for obtaining numerical solutions free from spatial truncation errors for all spatial cell sizes, one group and multigroup energy formulations in slab geometry problems with isotropic scattering and fixed-source. In the present paper, we extend the formulation of the numerical nodal method Spectral Deterministic Method (*SDM*) (OLIVA et al., 2016; OLIVA et al., 2018), to solve problems in one-dimension slab geometry, in the formulation of discrete ordinates (S_N), considering an arbitrary anisotropic scattering of order L in homogeneous and heterogeneous domains with a prescribed interior source.

The *SDM* method solves the analytic expression for the angular flux, instead of using the neutral particle balance equations together with the auxiliary equations for the cell averages fluxes. It uses the boundary conditions or estimates of the incoming angular fluxes in a node to determine the angular fluxes outgoing in the direction of the transport sweep. These estimates of the outgoing fluxes can vary from one region to the next if the material parameters between the regions are different.

This paper is organized as follows: Section 2 presents the spectral analysis of multigroup transport equations in discrete ordinates formulation. In Section 3, it is described the iterative methodology of the multigroup Spectral Deterministic Method (*SDM*) for heterogeneous geometry. Numerical results for one group linearly anisotropic scattering and an arbitrary anisotropic scattering of order L multigroup S_N problems are given in Section 4. A brief discussion of the results and suggestions for future work are presented in Section 5.

2. MATHEMATICAL PRELIMINARIES

This section exhibit the spectral analysis of neutron transport equation spatially and angular discretized in the multigroup formulation with anisotropic scattering of order L . Considering an arbitrary spatial grid Γ in the domain D , with length H as shown in Fig. 1, where each spatial node I_j has a width h_j and constant cross sections $\sigma_{s_{g' \rightarrow g}}^{(l)j}$ e $\sigma_{T_g}^j$



Source: The author, 2016.

Figure 1: Spatial grid Γ_j in a one-dimensional domain D with length H .

The multigroup S_N one-dimensional neutron transport equations with anisotropic scattering for a Cartesian geometry, defined in an arbitrary node Γ_j has the form

$$\mu_m \frac{d}{dx} \psi_{m,g}(x) + \sigma_{T_g}^j \psi_{m,g}(x) - \frac{1}{2} \sum_{l=0}^L (2l+1) P_l(\mu_m) \sum_{g'=1}^G \sigma_{S_{g' \rightarrow g}}^{(l)j} \sum_{n=1}^N P_l(\mu_n) \omega_n \psi_{n,g'}(x) + Q_g^j, \quad (1)$$

$l = 0 : L, m = 1 : N, g = 1 : G, x_{j-\frac{1}{2}} \leq x \leq x_{j+\frac{1}{2}}$

In Eq. (1), we used the standard notation, where $\psi_{m,g}(x)$ represents the angular flux of particles traveling in the discrete ordinates' direction μ_m for each energy group, N is the order of the Gauss-Legendre quadrature set (LEWIS and MILLER, 1993). The number of energy groups is represented by G and the geometric size of the problem by x , where $x \in \Gamma$. For each region analyzed, $\sigma_{T_g}^j$ describe the total macroscopic cross-section of the g -th group; which includes all possible interactions, $\sigma_{S_{g' \rightarrow g}}^{(l)j}$ is the l 'th-order component of the macroscopic g -th differential scattering cross section from group g' to group g . The remaining symbols in Eq. (1) are the order of scattering anisotropy L ; the Legendre polynomial of order l , $P_l(\mu_m)$; the angular weight for direction μ_m , ω_n and the constant isotropic source in the energy group g , Q_g^j .

The analytic nodal general solution of the system of equations (1) has the form:

$$\psi_{m,g}(x) = \psi_{m,g}^h(x) + \psi_{m,g}^p, \quad x \in \Gamma. \quad (2)$$

The superscript p denotes the particular solution with fixed-source and h indicates the homogeneous component of the local general solution, which satisfies the system of Eq. (1).

To determine the homogeneous solution $\psi_{m,g}^h$, we first set $Q_g^j = 0$, and then seek for elementary solutions of the form

$$\psi_{m,g}(x) = a_{m,g}(\vartheta) e^{-\frac{(x-x_{j-\frac{1}{2}})}{\vartheta}}, \quad m = 1 : N, g = 1 : G, x \in \Gamma_j, \quad (3)$$

where $x_{j-1/2}$ represents the left node-edge boundary of arbitrary node Γ_j . Substituting Eq. (3) in homogeneous part of Eq. (1), we obtain the eigenvalue problem

$$\sum_{g'=1}^G \sum_{n=1}^N \frac{\sigma_{Tg}^j}{\mu_m} \left[\delta_{m,n} \delta_{g'g} - \sum_{l=0}^L \frac{(2l+1) \sigma_{s_{g' \rightarrow g}}^{(l)j}}{2\sigma_{Tg}^j} P_l(\mu_m) P_l(\mu_n) \omega_n \right] a_{n,g'}(\vartheta) = \frac{1}{\vartheta} a_{m,g}(\vartheta),$$

$$m = 1:N, g = 1:G, \quad (4)$$

where

$$\delta_{a,b} = \begin{cases} 1 & \text{for } a = b, \\ 0 & \text{for } a \neq b, \end{cases}$$

is defined as the Kronecker delta.

This eigenvalue problem can be written in a compact notation as:

$$A\mathbf{a} = \frac{1}{\vartheta} \mathbf{a}, \quad (5)$$

where A is a real square matrix with $GN \times GN$ order.

The Eq. (5) constitutes the matrix representation of a homogeneous system of $GN \times GN$ linear equations with the unknown eigenvectors $\mathbf{a}(\vartheta_k)$, $k = 1:N$, which have GN components that correspond to the eigenvalues ϑ_k . Due to the symmetry of the Gauss-Legendre quadrature sets used in slab geometry with even N , the resulting polynomial equation has only even powers of ϑ_k , therefore the eigenvalues ϑ used will appear in pairs (BARROS and LARSEN, 1990). For fixed-source problems, which are the topic of the present work, these GN eigenvalues ϑ_k are real numbers.

The particular solution $\psi_{m,g}^p$ within the node Γ_j , is considered spatially constant for each energy group because of the sources Q_g^j in each region are assumed to be uniform and isotropic. Accordingly, by substitute the expression $\psi_{m,g}^p$ in Eq. (1) the particular solution takes the form

$$\sum_{g'=1}^G \sum_{n=1}^N \left(\sigma_{Tg}^j \delta_{m,n} \delta_{g'g} - \frac{1}{2} \sum_{l=0}^L (2l+1) \sigma_{s_{g' \rightarrow g}}^{(l)j} P_l(\mu_m) P_l(\mu_n) \omega_n \right) \psi_{m,g}^p = Q_g^j,$$

$$j = 1:J, \quad m = 1:N, \quad g = 1:G. \quad (6)$$

Therefore, the general solution $\psi_{m,g}(x)$ for the S_N equations for G energy groups in Γ_j given by Eq. (2), can be written in the following form:

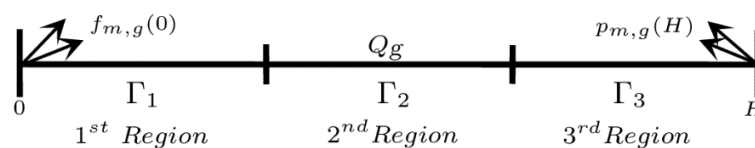
$$\psi_{m,g}(x) = \sum_{k=1}^{GN} \alpha_k a_{m,g}(\vartheta_k) e^{\frac{-(x-x_{j-\frac{1}{2}})}{\vartheta_k}} + \psi_{m,g}^p, \quad m = 1:N, \quad g = 1:G, \quad k = 1:GN, \quad x \in \Gamma_j \quad (7)$$

where α_k are arbitrary constants to be determined from the boundary conditions established for each problem to be analyzed and $\psi_{m,g}^p$ represent the particular solution obtained from Eq. (6).

3. THE MULTIGROUP SPECTRAL DETERMINISTIC METHOD (SDM) FOR HETEROGENEOUS PROBLEMS WITH L'TH ORDER ANISOTROPIC SCATTERING

In this section, the *Spectral Deterministic Method* (SDM) (OLIVA et al., 2016; OLIVA et al., 2018) and the iterative procedure to solve the neutron transport equation using the SDM method for heterogeneous problems is presented. Let us consider the heterogeneous slab represented in Fig. 1. This slab is divided into local sub-regions, which we refer to as spatial nodes Γ_j . In each node, the macroscopic cross sections, and the source Q_g^j are considered uniform, the spatial width of Γ_j is denoted by h_j . Therefore, every node can be considered as a homogeneous material region.

To begin, we consider Eq. (1), defined on the first homogeneous node Γ_1 of the spatial grid shown in Fig. 2,



Source: The author, 2016.

Figure 2: A heterogeneous slab of thickness H.

and considering

$$\psi_{m,g}(x) = \begin{cases} f_{m,g} & \text{for } x = 0, \\ p_{m,g} & \text{for } x = H, \end{cases}$$

as (prescribed) boundary conditions on the analyzed domain. The general solution $\psi_{m,g}(x)$, for Eq. (1) is given by Eq. (7). In order to solve Eq. (1), as shown in Section 2, we first obtain the numerical values for the eigenvectors $a_{m,g}(\vartheta_k)$ and the eigenvalues ϑ_k solving Eq. (4) in the analyzed material zone. The particular solution $\psi_{m,g}^p$ is obtained

solving Eq. (6), is considered spatially constant for each group within the node Γ_1 , once the sources are assumed to be uniform and isotropic in the analyzed region.

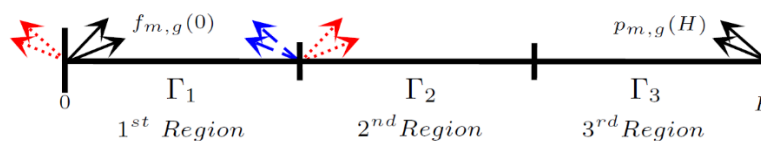
Now, can proceed to calculate the α_k parameters using Eq. (7) and the pre-established boundary conditions in node Γ_1 which are used as the initial estimates of the incoming angular fluxes in node left side, Fig. 3, black arrows. The initial estimates for the incoming angular fluxes in node right edge are shown with dashed arrows on Fig. 3.



Source: The author, 2016.

Figure 3: Incoming fluxes at node Γ_1 .

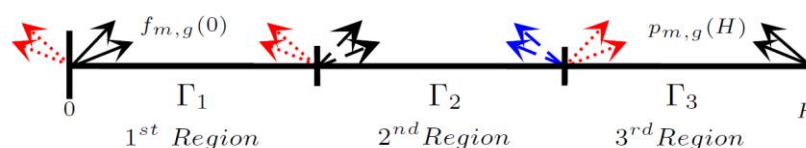
After obtained the α_k parameters can perform the calculations of the outgoing fluxes at node output (dotted arrows), Fig. 4, also using Eq. (7).



Source: The author, 2016.

Figure 4: Incoming and outgoing fluxes at node Γ_1 .

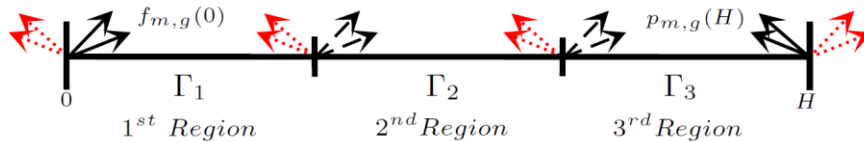
Applying continuity conditions, the outgoing angular fluxes of all energy groups at the right interface of the node Γ_1 , are used as the initial approximation for the incoming angular fluxes on the left side of the adjacent cell Γ_2 (dashed arrows emerging from the node Γ_1 , in Figure 5). The incoming angular fluxes from the right edge of the node Γ_2 continues as an initial approximation, (dashed arrows, in the right boundary of the node in Figure 5). With the incoming angular fluxes of the node Γ_2 , the parameters α_k of this node can be estimated and the outgoing angular fluxes calculated (dotted arrows in Figure 5).



Source: The author, 2016.

Figure 5: Incoming and outgoing fluxes at node Γ_2 .

Continuing the movement from left to right, using Eq. (7), to determine the parameters and outgoing angular fluxes on the remaining cells, for $\mu_m > 0$ and $\mu_m < 0$ directions, Fig. 6, until $x = H$. The incoming angular fluxes in Γ_3 cell right cell-edge are the prescribed boundary conditions.



Source: The author, 2016.

Figure 6: Incoming and outgoing fluxes in the heterogeneous slab.

This iterative process is performed until the prescribed stopping criterion of the maximum norm for the relative deviation between two consecutive estimates for the group scalar fluxes in the interface of the regions, Φ_g , does not exceed a pre-established value ε . This relative deviation is calculated using the equation

$$\max_{1 \leq j \leq (J+1)} \left| \frac{\Phi_{gj-\frac{1}{2}}^i - \Phi_{gj-\frac{1}{2}}^{i-1}}{\Phi_{gj-\frac{1}{2}}^i} \right| \leq \varepsilon, \quad g = 1:G, \quad (9)$$

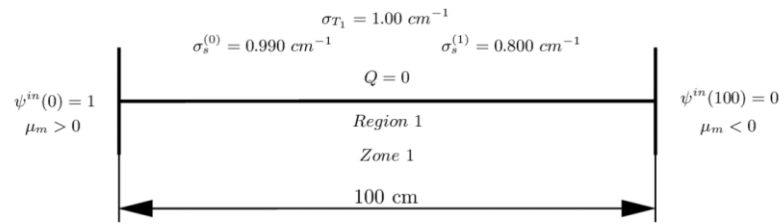
where $\left(\Phi_{gj-\frac{1}{2}}^i - \Phi_{gj-\frac{1}{2}}^{i-1} \right)$ is the difference between the scalar flux vector for the $(i-1)$ 'st previous iteration and the i 'th iteration which has just been executed, and

$$\Phi_g(x) = \frac{1}{2} \sum_{n=1}^N \psi_{n,g}(x) \omega_n, \quad (10)$$

gives us the scalar flux for each g 'th group.

4. NUMERICAL RESULTS

The first study case is a homogeneous slab with thickness $H = 100$ cm (DUDERSTADT and MARTIN, 1979). In this model problem, one group, no interior source, linearly anisotropic scattering transport problem is studied. The boundary conditions and macroscopic cross sections are shown in Fig. 7.



Source: The author, 2016.
Figure 7: Model-Problem No.1.

We solve this problem using the SDM method, with the S_2 , S_4 , S_8 Gauss-Legendre quadrature set with a varying number of spatial cells (BARROS and LARSEN, 1990). To compare our solution, we use the results reported by Barros and Larsen in (BARROS and LARSEN, 1990) for the SGF method, using one-cell block inversion and the results for the fine-mesh DD method is used as a reference in all cases. The cell interfaces scalar fluxes at $x = 0$, $x = 50$ and $x = 100$ are presented in Tables 1-3 for the SGF and SDM methods. In each table, the numbers in parentheses belong to the SGF method. For each case, the number of iterations required to achieve a stopping criterion of $\varepsilon = 10^{-7}$ is also listed.

The DD method reaches the fine-mesh at 400 nodes. For this case, we consider that mesh size meets our fine mesh parameters, because the values for scalar fluxes in the seventh decimal place do not have a significant variation and match with the established convergence criteria 10^{-7} .

Table 1: Scalar Flux ^a using S_2 angular quadrature at the homogeneous problem.

Number of Cells	$\Phi(0)$	$\Phi(50)$	$\Phi(100)$	Number of Iterations
400	8,1726E-01	1,6991E-02	1,2918E-04	376
100	8,1726E-01	1,6991E-02	1,2918E-04	376
	8,1726E-01	1,6991E-02	1,2918E-04	72
50	8,1726E-01	1,6991E-02	1,2918E-04	198
	8,1726E-01	1,6991E-02	1,2918E-04	64
20	8,1726E-01	1,6991E-02	1,2918E-04	85
	8,1726E-01	1,6991E-02	1,2918E-04	49
10	8,1726E-01	1,6991E-02	1,2918E-04	47
	8,1726E-01	1,6991E-02	1,2918E-04	36
4	8,1726E-01	1,6991E-02	1,2918E-04	25
	8,1726E-01	1,6991E-02	1,2918E-04	24
2	8,1726E-01	1,6991E-02	1,2918E-04	19
	8,1726E-01	1,6991E-02	1,2918E-04	19

^a Scalar Flux: $\text{cm}^{-2} \text{s}^{-1}$, ^b DD Method, ^c SDM Method, ^d SGF Method,
^e Read: 1.6991×10^{-2} .

Source: The author, 2016.

Table 2: Scalar Flux ^a using S_4 angular quadrature at the homogeneous problem.

Number of Cells	$\Phi(0)$	$\Phi(50)$	$\Phi(100)$	Number of Iterations
400	8,2226E-01	1,6538E-02	1,2353E-04	376
100	8,2226E-01	1,6538E-02	1,2353E-04	340
	8,2226E-01	1,6538E-02	1,2353E-04	83
50	8,2226E-01	1,6538E-02	1,2353E-04	179
	8,2226E-01	1,6538E-02	1,2353E-04	65
20	8,2226E-01	1,6538E-02	1,2353E-04	78
	8,2226E-01	1,6538E-02	1,2353E-04	45
10	8,2226E-01	1,6538E-02	1,2353E-04	43
	8,2226E-01	1,6538E-02	1,2353E-04	34
4	8,2226E-01	1,6538E-02	1,2353E-04	23
	8,2226E-01	1,6538E-02	1,2353E-04	23
2	8,2226E-01	1,6538E-02	1,2353E-04	18
	8,2226E-01	1,6538E-02	1,2353E-04	18

^a Scalar Flux: $\text{cm}^{-2} \text{s}^{-1}$, ^b DD Method, ^c SDM Method, ^d SGF Method,
^e Read: 1.6538×10^{-2} .

Source: The author, 2016.

Table 3: Scalar Flux ^a using S_8 angular quadrature at the homogeneous problem.

Number of Cells	$\Phi(0)$	$\Phi(50)$	$\Phi(100)$	Number of Iterations
400	8,2284E-01	1,6470E-02	1,2250E-04	376
100	8,2284E-01	1,6470E-02	1,2250E-04	329
	8,2284E-01	1,6470E-02	1,2250E-04	77
50	8,2284E-01	1,6470E-02	1,2250E-04	173
	8,2284E-01	1,6470E-02	1,2250E-04	61
20	8,2284E-01	1,6470E-02	1,2250E-04	75
	8,2284E-01	1,6470E-02	1,2250E-04	43
10	8,2284E-01	1,6470E-02	1,2250E-04	42
	8,2284E-01	1,6470E-02	1,2250E-04	32
4	8,2284E-01	1,6470E-02	1,2250E-04	23
	8,2284E-01	1,6470E-02	1,2250E-04	22
2	8,2284E-01	1,6470E-02	1,2250E-04	17
	8,2284E-01	1,6470E-02	1,2250E-04	17

^a Scalar Flux: $\text{cm}^{-2} \text{s}^{-1}$, ^b DD Method, ^c SDM Method, ^d SGF Method,
^e Read: 1.6470×10^{-2} .

Source: The author, 2016.

In the numerical results given in Tables 1 - 3, we observe that as the spatial cells become coarse, the numerical solution corresponding to the points at $x = 0$, $x = 50$, and $x = 100$ are the same for the SGF and SDM methods using the same quadrature sets. Thus, the solution to the problem using both methods is free from all spatial truncation errors. We also note that as the grid becomes coarser, the number of iterations necessary

to achieve the stopping criterion $\varepsilon = 10^{-7}$ becomes smaller for both methods.

The second study case, is a twenty energy groups ($G = 20$), anisotropic scattering of order $L = 10$, no interior source problem on a five material zones slab of thickness $H = 20$ cm (GARCIA and SIEWERT, 1983). According to (GARCIA and SIEWERT, 1983), the multilayer slab has an isotropic incoming in the fastest energy group ($g = 1$) on the left boundary for $\mu_m > 0$,

$$\psi_{m,g}(x) = \begin{cases} \psi_{m,g}(0) = \delta_{g,1}, \mu_m > 0, g = 1:20, \\ \psi_{m,g}(H) = 0, \mu_m < 0, g = 1:20. \end{cases} \quad (11)$$

The thickness of each material zone is defined by $h_j = (j + 1)cm, j = 1:5$. The fictitious group total macroscopic cross sections and the scattering cross sections are defined as

$$\sigma_{T_g}^j = \left(\frac{j+20}{21}\right)^5 \left[\frac{g}{10} - 0.15\delta_{g,5} - 0.15\delta_{g,10}\right], \quad g = 1:20 \quad (12)$$

and

$$\sigma_{S_g}^{(l)j} = \left(\frac{j+20}{21}\right)^5 \left[\frac{g'}{100(g-g'+1)}\right] h_{gg'}^l, \quad g = 1:g, \quad g = 1:20, \quad l = 0:10, \quad (13)$$

where

$$h_{gg'} = 0.7 - \frac{g+g'}{200} \quad (14)$$

In this problem were evaluated the albedos (A^*) and the transmission factors (B^*), which are defined by (GARCIA and SIEWERT, 1983) respectively as

$$A^* = 2 \sum_{\mu_m > 0} |\mu_n| \omega_n \psi_{n,g}^{out}(0), \quad g = 1:G, \quad (15)$$

where $\psi_{n,g}^{out}(0)$ represent the outgoing flux from the left boundary at $x = 0$ cm, and

$$B^* = 2 \sum_{\mu_m > 0} |\mu_n| \omega_n \psi_{n,g}^{out}(H), \quad g = 1:G, \quad (16)$$

where $\psi_{n,g}^{out}(H)$ represent the outgoing flux from the right boundary at $x = 20$ cm.

In Table 4 are shown the results for the S_{128} Gauss-Legendre quadrature set of the group's albedos, A_g^* , and the group transmission factors, B_g^* , together with the reference results generated with the F_N method in (GARCIA and SIEWERT, 1983). The prescribed stopping criterion is established in 10^{-10} (ASKEW, 1972).

Table 4: Group Albedos A_g^* and the Transmission Factors B_g^*

Groups	F_N method		SDM method	
	A_g^*	B_g^*	A_g^*	B_g^*
1	5.8809E-03	1.0453E-02	5.8853E-03	1.0453E-02
2	2.2791E-03	1.9993E-04	2.2805E-03	1.9993E-04
3	1.2939E-03	6.9012E-05	1.2947E-03	6.9016E-05
4	8.6280E-04	3.5393E-05	8.6338E-04	3.5398E-05
5	8.5170E-04	3.5350E-05	8.5213E-04	3.5355E-05
6	4.9662E-04	1.4899E-05	4.9698E-04	1.4904E-05
7	3.9706E-04	1.0716E-05	3.9737E-04	1.0721E-05
8	3.2763E-04	8.0863E-06	3.2789E-04	8.0907E-06
9	2.7671E-04	6.3202E-06	2.7694E-04	6.3246E-06
10	2.7956E-04	6.1271E-06	2.7977E-04	6.1320E-06
11	2.0989E-04	4.1837E-06	2.1008E-04	4.1881E-06
12	1.8491E-04	3.4892E-06	1.8508E-04	3.4935E-06
13	1.6476E-04	2.9545E-06	1.6492E-04	2.9587E-06
14	1.4814E-04	2.5332E-06	1.4829E-04	2.5374E-06
15	1.3423E-04	2.1953E-06	1.3436E-04	2.1994E-06
16	1.2242E-04	1.9200E-06	1.2254E-04	1.9240E-06
17	1.1229E-04	1.6927E-06	1.1241E-04	1.6967E-06
18	1.0352E-04	1.5029E-06	1.0363E-04	1.5068E-06
19	9.5859E-05	1.3428E-06	9.5965E-05	1.3467E-06
20	8.9125E-05	1.2064E-06	8.9225E-05	1.2103E-06

Source: The author, 2016.

Table 5, shows the relative deviation of the SDM results estimated as,

$$\varepsilon = \left(\frac{V_r - V}{V_r} \right) \times 100\%$$

where V_r is the reference value of the F_N method (ASKEW, 1972), and V is the quantity generated by the SDM method for the S_{128} Gauss-Legendre quadrature set.

Table 5: Relative deviation

ε	
A_g^*	B_g^*
7.5478E-02	3.1570E-03
6.0445E-02	7.9528E-04
6.3629E-02	6.1569E-03
6.7349E-02	1.3299E-02
5.0053E-02	1.4122E-02

7.3146E-02	3.2009E-02
7.7492E-02	4.3421E-02
7.9443E-02	5.4989E-02
8.3011E-02	6.9522E-02
7.4692E-02	7.9591E-02
8.9666E-02	1.0540E-01
9.2391E-02	1.2405E-01
9.5515E-02	1.4203E-01
1.0130E-01	1.6421E-01
9.7892E-02	1.8460E-01
1.0023E-01	2.0882E-01
1.0248E-01	2.3591E-01
1.0417E-01	2.6266E-01
1.1013E-01	2.8872E-01
1.1181E-01	3.2326E-01

Source: The author, 2016.

As can be seen in Table 5, the relative deviation of the SDM results are minimal, $|\varepsilon| < 1\%$, concerning the reference results of (GARCIA and SIEWERT, 1983). The existing differences are due to finite computational arithmetic.

5. DISCUSSION

An analytic coarse-mesh numerical method, *SDM*, for computational modeling of neutral particles in slab geometry with an arbitrary degree of anisotropy have been presented. The multigroup *SDM* discretization scheme preserves the general analytic solution of the multigroup S_N equation in each spatial node converges to numerical results that are continuous across each node interface and satisfies the external boundary conditions whether the mesh size order or quadrature set used. The resulting numerical solution is free of spatial truncation errors, same as the *SGF* method while the fine-mesh *DD* method solution is not.

The iterative numerical strategy used for this method, when compared with the *DD* method, allows reaching the convergence faster. The iteration numbers required to converge the same grid problem with the same quadrature sets for both the *SGF* method

with the one-cell block inversion and the SDM are equivalent, despite our route or iterations are substantially different than the transport sweeps employed by the DD , SGF and F_N methods. Another advantage presented for this method when compared with the other methods used in this paper is the simplification for the solution algorithm of the discretized equations of the S_N problems. The SDM method exhibit the same accuracy, stability, and consistency presented by the other methods DD , SGF and F_N for both study cases analyzed.

6. FINAL CONSIDERATIONS

Based on the results presented in the study, we conclude with the comment that the SDM method could be applied in multidimensional geometries, due to the solution algorithm simplification for slab geometry problems. Our goal is extending the study and apply the present analytical-numerical method to X, Y geometry multigroup problems with anisotropic scattering and multidimensional calculations in rectangular geometry, by using transverse integration with prescribed approximations for the transverse leakage terms (BARROS, 1990). For example, if we consider an S_N problem in a two-dimensional geometry (x, y) , as in the SGF method, we could integrate the S_N equation in the x - direction and then over the y direction, driving us to the transverse leakage terms to the source side to obtain a system of two one dimensional S_N equations (BARROS, 1990), and then it can be solved using the same methodology used in this paper for one dimensional. If successful, similar ideas, can also be implemented for three-dimensional problems.

We plan to report on the results of the analytic-numerical methods after they have been implemented and thoroughly tested.

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